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Rotations associated with Lorentz boosts

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Abstract

It is possible to associate two angles with two successive non-collinear Lorentz boosts. If one boost is applied after the initial boost, the result is the final boost preceded by a rotation called the Wigner rotation. The other rotation is associated with Wigner's $O(3)$-like little group. These two angles are shown to be different. However, it is shown that the sum of these two rotation angles is equal to the angle between the initial and final boosts. This relation is studied for both low-speed and high-speed limits. Furthermore, it is noted that the two-by-two matrices which are under the responsibility of other branches of physics can be interpreted in terms of the transformations of the Lorentz group, or vice versa. Classical ray optics is mentioned as a case in point.

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1. Introduction

The Wigner rotation is known as a by-product of two successive Lorentz boosts in special relativity. The earliest manifestation of the Wigner rotation is the Thomas precession which we observe in atomic spectra. Thomas formulated this problem 13 years before the appearance of Wigner's 1939 paper [1, 2]. The Thomas effect in nuclear spectroscopy is mentioned in Jackson's book on electrodynamics [3]. Indeed, the Wigner rotation is the key issue in many branches of physics involving Lorentz boosts [4].

The Wigner rotation is not restricted to relativistic kinematics. It appears in physical processes whose underlying mathematical language includes the Lorentz group. Berry's phase is a case in point [5, 6]. This branch of physics deals with a physical system which gains a phase angle when it comes back to the original state after undergoing a series of transformations. If the transformations include those of a group isomorphic to the Lorentz group, the Wigner rotation plays a role [7].

Recently, the Lorentz group has become an important scientific language in both quantum and classical optics. The theory of squeezed states is a representation of the Lorentz group [8, 9]. Optical instruments are ubiquitous in modern physics, and they are based on classical ray optics. It is gratifying to observe that the Lorentz group, through its two-by-two representation, is the basic underlying scientific language for ray optics, including polarization optics [10], interferometers [11], lens optics [12, 13], laser cavities [14] and multi-layer optics [15].

It is possible to perform mathematical operations of the Lorentz group by arranging optical instruments. For instance, the group contraction is one of the most sophisticated operations in the Lorentz group, but it has been shown recently that this can be achieved through a focusing process in one-lens optics [13]. Since there are many mathematical operations in optical sciences corresponding to Lorentz boosts, the Wigner rotation becomes one of the important issues in classical and quantum optics.

If we perform two Lorentz boosts in different directions, the result is not a boost, but is a boost preceded or followed by a rotation. This rotation is commonly known as the Wigner rotation. However, if we trace the origin of this word, Wigner introduced the rotation subgroup of the Lorentz group whose transformations leave the four momentum of a given particle invariant in its rest frame. The rotation can however change the direction of its spin. Indeed, Wigner introduced the concept of 'little group' to deal with this type of problem. Wigner's little group is the maximum subgroup of the Lorentz group whose transformations leave the four momentum of the particle invariant. If the particle is moving, we can go to the Lorentz frame where it is at rest, perform a rotation without changing the momentum and then come back to the original Lorentz frame. These transformations leave the momentum invariant. We shall hereafter call this little-group rotation 'WLG rotation'.

The question then is whether the Wigner rotation, as understood in the literature, is the same as the WLG rotation. This question was raised by Han *et al* in their paper on Thomas precession and gauge transformations, but they have not made any attempt to clarify this issue [16]. The present authors raised this question again in their paper on laser cavities [14]. They first noted that the two-by-two matrix formulation of lens optics is a representation of the Lorentz group, and then showed that the light beam performs one little-group rotation as it goes through one cycle in the cavity. Then they showed that the Wigner rotation and the WLG rotation are different, but those rotation angles were related for the special case of the Thomas precession.

The purpose of this paper is to establish the same relation for the most general case. We establish the difference between those two angles, and then show that they satisfy a complementary relation. In spite of the simplicity in concept, the calculations of these angles are not trivial.

Every relativistic problem has two important limits. One is the non-relativistic limit, and the other is the light-like limit where the momentum of the particles becomes infinitely large. We also study these angles and their relation for the two limiting cases.

We note that the $SL(2, C)$, the group of unimodular two-by-two matrices, is the universal covering group of the Lorentz group, having the same algebraic property as that of the four-by-four representation of the Lorentz group. Although, for completeness we have included the expressions of the four-by-four transformation matrices, needless to say, their two-by-two counterparts can be expressed in a much more compact way. Furthermore, and more importantly, within the $SL(2, C)$ formalism these matrix calculations can be applied to the two-by-two beam transfer matrices and the two-by-two lens matrices in classical ray optics. Indeed, our basic motivation for this paper came from our experience in ray optics. Thus, the group $SL(2, C)$ provides not only a topological base for the Lorentz group, but also concrete calculational tools for various branches of physics.

In section 2, we consider two different rotations associated with two successive non-collinear Lorentz boosts. One is the Wigner rotation, and the other is the rotation associated

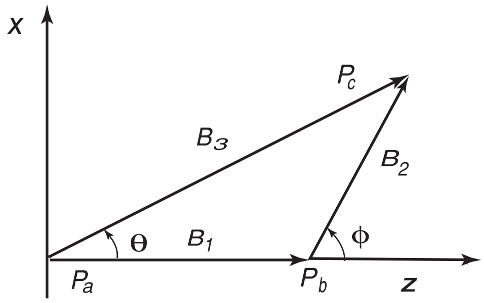

Figure 1. Two successive Lorentz boosts. Let us start from a particle at rest. If we make boost B_1 along the z direction and another B_2 along the direction with makes an angle of ϕ with the z direction, the net result is not B_3, but B_3 preceded by a rotation. This rotation is known as the Wigner rotation.

with Wigner's little group. It is shown that the addition of these two angles is equal to the angle between the direction of the first boost and the final boost. In section 3, using the two-by-two matrices, we explicitly calculate those angles in terms of the parameters of the initial Lorentz boosts. In section 4, we give some illustrative examples to show the dependence of the angles on the boost parameters. In section 5, we explain how special relativity and ray optics find a common mathematical ground through their two-by-two matrix formalism.

2. Two different angles

In the literature, the Wigner rotation comes from two successive Lorentz boosts performed in different directions. If we boost along the z axis first and then make another boost along the direction which makes an angle ϕ with the z axis on the zx plane as shown in figure 1, the result is another Lorentz boost preceded by a rotation. This rotation is known as the Wigner rotation in the literature.

The rotation matrix which performs a rotation around the y axis by angle ϕ is

$$R(\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

and its inverse is $R(-\phi)$. For convenience, here and in the sequel we have adopted the ordering of the coordinate system as (t, z, x, y).

The boost matrix requires two parameters. One is the boost parameter, and the other is the angle specifying the direction. We shall use the notation

$$B(\phi, \eta) \tag{2}$$

as the matrix performing a boost along the direction which makes an angle of ϕ with the z axis in the zx plane. Accordingly, we have $B(0, \eta_1)B(0, \eta_2) = B(0, \eta_1 + \eta_2)$. The boost matrix along the z direction takes the form

$$B(0, \eta) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

If this boost is made along the ϕ direction, the matrix is

$$B(\phi, \eta) = R(\phi)B(0, \eta)R(-\phi), \tag{4}$$

and its inverse is $B(\phi, -\eta)$.

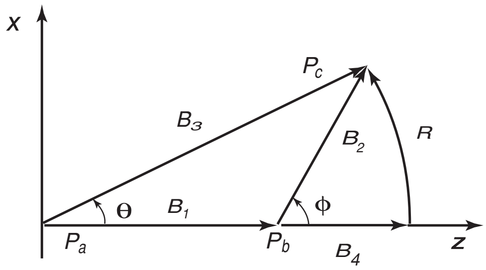

Figure 2. Closed Lorentz boosts: initially, a massive particle is at rest with its four-momentum P_a. The first boost B_1 brings P_a to P_b. The second boost B_2 transforms P_b to P_c. The inverse of the third boost, B_3^{-1} brings P_c back to P_a. The particle is again at rest. The net effect is a rotation around the axis perpendicular to the plane containing these three transformations. We may assume for convenience that P_b is along the z axis, and P_c is in the zx plane. The rotation is then made around the y axis. An alternative way which transforms P_b to P_c is first to boost P_b by B_4 (i.e., by $B(0, \xi - \eta)$), and then to rotate it by $R(\theta)$. Finally, the second 'loop' is closed by B_2^{-1}, which brings P_c back to P_b.

Let us start with a massive particle at rest whose four-momentum is

$$P_a = (m, 0, 0, 0), \tag{5}$$

where m is the particle mass. If we apply the first boost matrix $B(0, \eta)$ to the four-momentum it becomes

$$P_b = m(\cosh\eta, \sinh\eta, 0, 0). \tag{6}$$

The successive application of the second boost $B(\phi, \lambda)$ to the four-momentum P_b will be the same as the application of the third boost $B(\theta, \xi)$ to P_a. Then the four-momentum becomes

$$P_c = m(\cosh\xi, (\sinh\xi)\cos\theta, (\sinh\xi)\sin\theta, 0). \tag{7}$$

The kinematics of these transformations is illustrated in figure 1, where the matrices $B(0, \eta)$, $B(\phi, \lambda)$ and $B(\theta, \xi)$ correspond to B_1, B_2 and B_3, respectively.

Then, we can consider the successive boosts

$$B(\theta, -\xi)B(\phi, \lambda)B(0, \eta). \tag{8}$$

If this matrix is applied to P_a of equation (5), it brings back to P_a. This means that the net effect is a rotation $R(\omega)$, which does not change the four-momentum of the particle in its rest frame. This aspect is commonly written in the literature as

$$B(\phi, \lambda)B(0, \eta) = B(\theta, \xi)R(\omega). \tag{9}$$

The product of the two boost matrices appears to be one boost matrix on the right-hand side in figure 1, but there must be a rotation matrix $R(\omega)$ to complete the mathematical identity. This rotation is known as the Wigner rotation in the literature:

$$R(\omega) = B(\theta, -\xi)B(\phi, \lambda)B(0, \eta). \tag{10}$$

Let us consider a different transformation to obtain P_c from P_b. We can first boost the system by $B(0, \xi - \eta)$, and rotate it by $R(\theta)$. The boost along the same direction does not change the helicity of the particle. The rotation $R(\theta)$ is also a helicity preserving transformation. This route is illustrated in figure 2. Helicity-conserving transformations have been discussed extensively in the literature [14, 17].

There are now two different ways of obtaining P_c from P_b. If we choose the second route, and come back using $B(\phi, -\lambda)$, the net effect is

$$D(\eta, \lambda, \phi) = B(\phi, -\lambda)[R(\theta)B(0, \xi - \eta)]. \tag{11}$$

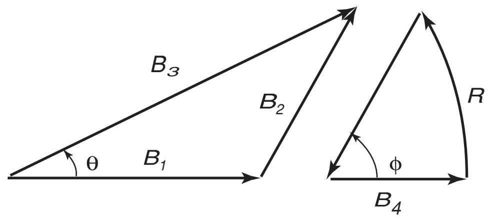

Figure 3. Addition of the angles. This figure consists of figure 1 and the kinematics corresponding to the D matrix of equation (11). This figure also illustrates the addition rule of equation (16).

This transformation leaves the four-momentum P_b given in equation (6) invariant. This 'loop' transformation is illustrated in figure 3.

This is not the only way to leave the given the four-momentum unchanged. If we apply the boost $B(0, -\eta)$ to the four-momentum P_b of equation (6), the result would be the four-momentum P_a of equation (5). This is the four-momentum of the particle at rest. This four momentum is invariant under three-dimensional rotations. This is precisely what Wigner observed in defining the $O(3)$-like rotation group for massive particles [2]. After performing a rotation which leaves P_a invariant, we can boost the momentum back to P_b by applying $B(0, \eta)$. The net effect is

$$B(0, \eta)R(\alpha)B(0, -\eta). \tag{12}$$

This is the original definition of Wigner's little group which leaves P_b invariant. The rotation matrix $R(\alpha)$ represents a three-dimensional rotation matrix.

We now demand that the little-group transformation of equation (12) be the same as the D matrix of equation (11). Then,

$$B(0, \eta)R(\alpha)B(0, -\eta) = B(\phi, -\lambda)R(\theta)B(0, \xi - \eta). \tag{13}$$

This determines the angle α as

$$R(\alpha) = B(0, -\eta)B(\phi, -\lambda)R(\theta)B(0, \xi). \tag{14}$$

This is the WLG rotation angle as defined in section 1.

Let us next consider the product $R(\omega)R(\alpha)$, where $R(\omega)$ and $R(\alpha)$ are from equations (10) and (14), respectively. Then

$$R(\omega)R(\alpha) = R(\theta), \tag{15}$$

which leads to

$$\alpha + \omega = \theta. \tag{16}$$

It is interesting to note that the above relation does not depend on the direction of the $B(\theta, \xi)$, nor does on the boost parameters η and λ.

The purpose of this paper is to study consequences of the above relation.

3. Computation of the rotation angles

In this section, we compute both Wigner rotation and WLG rotation angles. The two-by-two representation of the rotation matrix corresponding to the four-by-four expression of equation (1) is

$$R(\phi) = \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{17}$$

while the boost matrix given in equation (3) becomes

$$B(0, \eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{18}$$

Let us use $B(\phi, \eta)$ for the boost along the direction which makes an angle ϕ with the z axis. Then it takes the form

$$\begin{pmatrix} \cosh(\eta/2) + (\cos\phi)\sinh(\eta/2) & (\sin\phi)\sinh(\eta/2) \\ (\sin\phi)\sinh(\eta/2) & \cosh(\eta/2) - (\cos\phi)\sinh(\eta/2) \end{pmatrix}. \tag{19}$$

Using these two-by-two expressions, we can complete all the computations for the transformation matrices given in section 2.

Let us go to the calculation of the Wigner rotation angle defined in equation (9). We can compute ξ, θ and ω in terms of η, λ and ϕ, by requiring that the right-hand side of equation (10) be a rotation matrix [18, 19]. The result of this calculation is

$$\begin{aligned} \cosh\xi &= \cosh\eta\cosh\lambda + \sinh\eta\sinh\lambda\cos\phi, \\ \tan\theta &= \frac{\sin\phi[\sinh\lambda + \tanh\eta(\cosh\lambda - 1)\cos\phi]}{\sinh\lambda\cos\phi + \tanh\eta[1 + (\cosh\lambda - 1)\cos^2\phi]}, \\ \tan\omega &= \frac{2(\sin\phi)[\sinh\lambda\sinh\eta + C_-\cos\phi]}{C_+ + C_-\cos(2\phi) + 2\sinh\lambda\sinh\eta\cos\phi}, \end{aligned} \tag{20}$$

with

$$C_\pm = (\cosh\lambda \pm 1)(\cosh\eta \pm 1). \tag{21}$$

As for the angle α, we first compute the boost parameter β of B_4 in terms of η, λ and ϕ as

$$\tanh\beta = \frac{f - \tanh\eta(1 + \tanh\eta\tanh\lambda\cos\phi)}{(1 + \tanh\eta\tanh\lambda\cos\phi) - f\tanh\eta}, \tag{22}$$

and then obtain the D matrix of equation (11) which takes the form

$$D(\eta, \lambda, \phi) = \begin{pmatrix} [(f+g)/2f]^{1/2} & [h_+(f-g)/2f]^{1/2} \\ [h_-(f-g)/2f]^{1/2} & [(f+g)/2f]^{1/2} \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} f &= \frac{\sqrt{(\cosh\eta\cosh\lambda + \sinh\eta\sinh\lambda\cos\phi)^2 - 1}}{\cosh\eta\cosh\lambda}, \\ g &= \tanh\eta + \tanh\lambda\cos\phi, \qquad h_\pm = \frac{1 \pm \tanh\eta}{1 \mp \tanh\eta}. \end{aligned} \tag{24}$$

The four-by-four counterpart of $D(\eta, \lambda, \phi)$ is of the form

$$\begin{pmatrix} [f\cosh^2\eta - g\sinh^2\eta]/f & n(g-f)/f & -\kappa/f & 0 \\ -n(g-f)/f & [-f\sinh^2\eta + g\cosh^2\eta]/f & -s/f & 0 \\ -\kappa/f & s/f & g/f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where

$$\kappa = \tanh\eta\tanh\lambda\sin\phi, \qquad n = \sinh\eta\cosh\eta, \qquad s = \tanh\lambda\sin\phi. \tag{26}$$

On the other hand, the left-hand side of equation (11) is $B(0, \eta)R(\alpha)B(0, -\eta)$, which takes the form

$$\begin{pmatrix} \cos(\alpha/2) & -e^{\eta/2}\sin(\alpha/2) \\ e^{-\eta/2}\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}. \tag{27}$$

Now, in view of equation (13), we can calculate the angle α by equating equations (23) and (27). The result is

$$\tan\alpha = \frac{\tanh\lambda\sin\phi}{\sinh\eta + \cosh\eta\tanh\lambda\cos\phi}. \tag{28}$$

We can check the addition law given in equation (16) by computing

$$\tan(\alpha+\omega) = \frac{\tan\alpha + \tan\omega}{1 - (\tan\alpha)\tan\omega}. \tag{29}$$

After completion of this calculation using $\tan\omega$ and $\tan\alpha$ of equations (20) and (28) respectively, we end up with $\tan\theta$ of equation (20).

In terms of the velocity of the particle, $\tanh\eta = v/c$. This means that $v = c\eta$ in the slow-speed limit. If the particle speed approaches the speed of light, $\tanh\eta$ becomes 1. Let us consider the velocity additions in both cases. If η and λ are both small, the expressions in equation (20) become

$$\begin{aligned} &\xi^2 = \eta^2 + \lambda^2 + \eta\lambda\cos\phi, \qquad \tan\theta = \frac{\lambda\sin\phi}{\eta + \lambda\cos\phi}, \\ &\alpha = \theta, \qquad \omega = 0. \end{aligned} \tag{30}$$

These expressions are consistent with the addition rules of non-relativistic kinematics. The Wigner rotation does not exist because $\omega = 0$.

If η and λ are small, the system becomes the non-relativistic case. If η becomes infinitely large, we are dealing with light-like particles. In the limit of large η we have

$$\begin{aligned} &\xi = \eta + \ln(\cosh\lambda + \sinh\lambda\cos\phi), \\ &\tan\theta = \frac{\sin\phi[\sinh\lambda + (\cosh\lambda - 1)\cos\phi]}{\sinh\lambda\cos\phi + [1 + (\cosh\lambda - 1)\cos^2\phi]}, \\ &\alpha = 0, \qquad \omega = \theta. \end{aligned} \tag{31}$$

As for the D matrix of equation (23), it becomes

$$\begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}, \tag{32}$$

in the limit of small η and λ. This matrix represents a rotation by an angle α around the y axis. This form is consistent with the expressions given in equation (27).

Let us go back to the original definition of Wigner's little group for massive particles. For a given massive particle, moving along the z direction, we can bring the particle to its rest frame. Then we can perform a rotation without changing the four-momentum of the particle. However, the direction of its spin changes. We can bring back the particle to its original momentum by applying a boost matrix. This is what is happening in equation (12). If the amount of boost is very small, the little-group transformation is a rotation as given in equation (32).

For massless particles, it is not possible to bring the particle to its rest frame. The best we can do is to align the z axis along the direction of the momentum. In his original paper [2], Wigner observed that the subgroup of the Lorentz group which dictates the internal spacetime symmetry is locally isomorphic to the two-dimensional Euclidean group, with one rotational and two translational degrees of freedom. The rotational degree of freedom corresponds to the helicity, but the translation-like degrees were left unexplained.

Let us look at the D matrix of equation (23). When η becomes very large, and $\tanh\eta$ approaches 1, this matrix becomes

$$D(\lambda,\phi) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tag{33}$$

where

$$u = \frac{2\tanh\lambda\sin\phi}{1+\tanh\lambda\cos\phi}. \tag{34}$$

Similarly, when $\tanh\eta$ approaches to 1, the D matrix of equation (25) becomes

$$D = \begin{pmatrix} 1+u^2/4 & -u^2/2 & -u & 0 \\ u^2/2 & 1-u^2/2 & -u & 0 \\ -u & u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

This expression was given in Wigner's original paper [2], and corresponds to one of the translation-like transformations for the massless particle, but its physical interpretation as a gauge transformation was first given by Janner and Janssen [20]. Indeed, this matrix has had a stormy history [21–23], and its full story had not been told until 1990 when Kim and Wigner presented a cylindrical picture of the $E(2)$-like little group for massless particles [25]. This little group as a generator of gauge transformations is also an interesting subject in general relativity [24].

Furthermore, it is interesting to see that the expression of the D matrix can be obtained as a large-η limit of the Lorentz-boosted rotation of equation (27). This is a procedure known as the group contraction which Inönü and Wigner introduced to physics in 1953 [26]. In their paper, Inönü and Wigner considered a two-dimensional plane tangent to a sphere, and observed that a small area on the spherical surface can be regarded as a two-dimensional plane with the two-dimensional Euclidean symmetry. Indeed, the Inönü–Wigner contraction is the contraction of the rotation group $O(3)$ to the two-dimensional Euclidean group.

Since the symmetry groups for massive and massless particles are locally isomorphic to the rotation and Euclidean groups respectively, it was expected that the symmetry group of massless particle could be obtained through a contraction procedure. This aspect also has a history [27, 28], but the problem had not been completely clarified when Kim and Wigner in 1990 introduced a cylindrical symmetry for massless particles [25]. The question was that there are two-translational degrees of freedom while there is only one gauge degree of freedom.

4. Illustrative examples

The calculations of section 3 become simpler if the angle ϕ takes a special value. If this angle is such that the boost parameter ξ remains the same as η, this transformation is responsible for Thomas precession. For this simpler case, the addition law $\theta = \alpha + \omega$ was noted in our earlier paper [14]. The formulae of equations (20) and (28) become

$$\begin{aligned} &\cosh\xi = \cosh\eta, \qquad \tan\theta = \tan\theta, \\ &\tan\alpha = \frac{2\sin\theta\cosh\eta}{\sinh^2\eta + (1+\cosh^2\eta)\cos\theta)}, \\ &\tan\omega = \frac{\sin\theta[\cos\theta(\cosh\eta-1)^2+\sinh^2\eta}{\cos\theta[\cos\theta[(\cosh\eta-1)^2+\sinh^2\eta]+2\cosh\eta}. \end{aligned} \tag{36}$$

In our earlier paper [14], we calculated α and ω in terms θ and η, instead of λ and η.

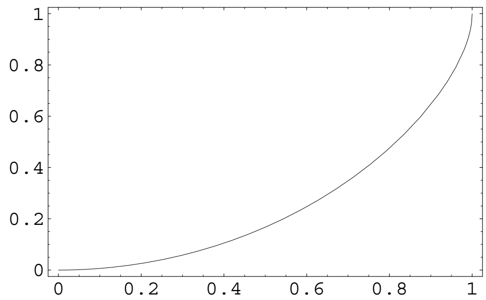

Figure 4. The ratio of the angle ω to the angle θ as a function of $\tanh\eta$, which becomes one as η becomes very large. The ratio is zero at $\eta = 0$, while it becomes one as η approaches infinity. This was expected from the limiting cases discussed at the end of section 3.

If the angle ϕ is 90°, the expressions of equations (20) and (28) also become simpler, and the kinematics becomes quite transparent [29]. The angles are

$$\cosh\xi = \cosh\eta\cosh\lambda, \qquad \tan\theta = \frac{\sinh\lambda}{\tanh\eta},$$
$$\tan\alpha = \frac{\tanh\lambda}{\sinh\eta}, \qquad \tan\omega = \frac{\sinh\lambda\sinh\eta}{\cosh\eta + \cosh\lambda}. \tag{37}$$

We can now plot the above expressions as η goes from zero to infinity, or as $\tanh\eta$ goes from zero to one, for a given value of λ. Let us try the case with $\lambda = \eta$. Then the expressions become

$$\cosh\xi = \cosh^2\eta, \qquad \tan\theta = \cosh\eta,$$
$$\tan\alpha = \frac{1}{\cosh\eta}, \qquad \tan\omega = \frac{\sinh\eta\tanh\eta}{2}. \tag{38}$$

In terms of $\tanh\eta$,

$$\cosh\xi = \frac{1}{1-\tanh^2\eta}, \qquad \tan\theta = \frac{1}{\sqrt{1-\tanh^2\eta}},$$
$$\tan\alpha = \sqrt{1-\tanh^2\eta}, \qquad \tan\omega = \frac{\tanh^2\eta}{2\sqrt{1-\tanh^2\eta}}. \tag{39}$$

If we plot the angle θ against $\tanh\eta$, it starts with 45° at $\eta = 0$. The angle monotonically increases to 90° as $\tanh\eta$ reaches 1. We can also plot α and ω to appreciate the addition rule given in equation (16).

In figure 4, the ratio of the angle ω to the angle θ is plotted as a function of $\tanh\eta$, which becomes one as η becomes very large. The ratio is zero at $\eta = 0$, while it becomes one as η approaches infinty.

5. Physics of two-by-two matrices

According to Eugene Wigner, quantum mechanics is the physics of Fourier transformations, and special relativity is the physics of Lorentz transformations.

In our recent papers, we formulated classical ray optics in terms of the two-by-two matrix representation of the Lorentz group, meaning that special relativity and ray optics have found a common mathematical formulation. It was noted that optical instruments can serve as analogue computers for special relativity through the use of those two-by-two matrices. Most of the calculations done in this paper, particularly the group contraction mentioned in section 3, can be carried out by optical instruments [13].

Furthermore, we have shown that in a laser cavity the Wigner little-group angle can be associated to the beam going through one cycle [14]. This is a simple cavity consisting of two identical concave mirrors with radii R and are separated by a distance d. The ABCD matrix of such a cavity can be expressed as

$$EC^2E^{-1} \tag{40}$$

where

$$C = \begin{pmatrix} 1-\frac{d}{R} & 1-\frac{d}{2R} \\ \frac{-2d}{R} & 1-\frac{d}{R} \end{pmatrix} \tag{41}$$

and

$$E = \begin{pmatrix} \sqrt{d} & \frac{-\sqrt{d}}{2} \\ 0 & \frac{1}{\sqrt{d}} \end{pmatrix}, \tag{42}$$

with $(2R > d)$, which takes care of the stability condition.

It is now possible to identify the core matrix C with equation (27). If the boost parameter ξ is equal to the boost parameter η as in section 4, then the WLG angle α, the Lorentz group angle θ and the Wigner angle ω can be expressed in terms of the physically measurable quantities R and d of the laser cavity as

$$\tan(\alpha/2) = \frac{a}{R-d}, \tag{43}$$

$$\tan(\theta/2) = \frac{b}{2R}, \tag{44}$$

$$\tan(\omega/2) = \frac{(R-d)b-2Ra}{2R(R-d)+ab}, \tag{45}$$

where

$$a = [d(2R-d)]^{\frac{1}{2}}, \qquad b = [a(2R+3d)]^{\frac{1}{2}}. \tag{46}$$

Indeed, the motivation of this work is substantially based on the results of the papers written earlier by the present authors on ray optics.

Coherent and squeezed states in quantum optics can be formulated in terms of Wigner functions defined in two-dimensional phase space and linear canonical transformations [9]. Many physical theories are formulated as two-level problems. Most of the soluble models in physics take the form of coupled harmonic oscillators. Needless to say, all those diverse areas of physics are based on the mathematics of two-by-two matrices.

Einstein introduced his special relativity one hundred years ago. This theory of course revolutionized our understanding of space and time, and thereby introduced to physics a mathematical device called the Lorentz group. Through its two-by-two matrix representation, the Lorentz group is a very powerful instrument in theoretical physics.

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